The Lieb-Liniger Model as a Limit of Dilute Bosons in Three Dimensions

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Abstract

We show that the Lieb-Liniger model for one-dimensional bosons with repulsive δ-function interaction can be rigorously derived via a scaling limit from a dilute three-dimensional Bose gas with arbitrary repulsive interaction potential of finite scattering length. For this purpose, we prove bounds on both the eigenvalues and corresponding eigenfunctions of three-dimensional bosons in strongly elongated traps and relate them to the corresponding quantities in the Lieb-Liniger model. In particular, if both the scattering length a and the radius r of the cylindrical trap go to zero, the Lieb-Liniger model with coupling constant $g \sim a/r^2$ is derived. Our bounds are uniform in g in the whole parameter range $0 \leq g \leq \infty$, and apply to the Hamiltonian for three-dimensional bosons in a spectral window of size $\sim r^{-2}$ above the ground state energy.

1 Introduction

Given the success of the Lieb-Liniger model [12, 11], both as a toy model in statistical mechanics and as a concrete model of dilute atomic gases in strongly elongated traps, it is worth investigating rigorously its connection to three-dimensional models with genuine particle interactions. A first step in this direction was taken in [17, 16], where it was shown that in an appropriate scaling limit the ground state energy of a dilute three-dimensional Bose gas is given by the ground state energy of the Lieb-Liniger model. The purpose of this paper is to extend this result to excited energy eigenvalues and the corresponding eigenfunctions.

The Lieb-Liniger model has recently received a lot of attention as a model for dilute Bose gases in strongly elongated traps [20, 22, 6, 9, 2, 16, 21, 23, 3]. Originally introduced as a toy model of a quantum many-body system, it has now become relevant for the description of actual quasi one-dimensional systems. The recent

advances in experimental techniques have made it possible to create such quasi one-dimensional systems in the laboratory [5, 18, 24, 10, 25, 19, 7]. These provide a unique setting for studying matter under extreme conditions where quantum effects dominate.

The Lieb-Liniger model describes n non-relativistic bosons in one spatial dimension, interacting via a δ-function potential with strength $g \geq 0$. In appropriate units, the Hamiltonian is given by

$$H_{1\text{d}}^{n,\ell,g} = \sum_{i=1}^{n} \left(-\partial_i^2 + \ell^{-2} V^{\|}(z_i/\ell) \right) + g \sum_{1 \leq i < j \leq n} \delta(z_i - z_j) \,. \tag{1.1}$$

Here, we use the notation $\partial_i = \partial/\partial z_i$ for brevity. The trap is represented by the potential $V^{\|}$, which is assumed to be locally bounded and tend to infinity as $|z| \to \infty$. The scaling parameter ℓ is a measure of the size of the trap. Instead of the trap potential $V^{\|}$, one can also confine the system to an interval of length ℓ with appropriate boundary conditions; in fact, periodic boundary conditions were considered in [12, 11].

The Lieb-Liniger Hamiltonian $H_{1\text{d}}^{n,\ell,g}$ acts on totally symmetric wavefunctions $\phi \in L^2(\mathbb{R}^n)$, i.e., square-integrable functions satisfying $\phi(z_1, \ldots, z_n) = \phi(z_{\pi(1)}, \ldots, z_{\pi(n)})$ for any permutation π. In the following, all wavefunctions will be considered symmetric unless specified otherwise.

In the case of periodic boundary conditions on the interval $[0, \ell]$, Lieb and Liniger have shown that the spectrum and corresponding eigenfunctions of $H_{1\text{d}}^{n,\ell,g}$ can be obtained via the Bethe ansatz [12]. In [11] Lieb has specifically studied the excitation spectrum, which has an interesting two-branch structure. This structure has recently received a lot of attention [9, 3] in the physics literature. Our results below show that the excitation spectrum of the Lieb-Liniger model is a genuine property of dilute three-dimensional bosons in strongly elongated traps in an appropriate parameter regime.

In the following, we shall consider dilute three-dimensional Bose gases in strongly elongated traps. Here, *dilute* means that $a\varrho^{1/3} \ll 1$, where a is the scattering length of the interaction potential, and ϱ is the average particle density. *Strongly elongated* means that $r \ll \ell$, where r is the length scale of confinement in the directions perpendicular to z. We shall show that, for fixed n and ℓ, the spectrum of a three dimensional Bose gas in an energy interval of size $\sim r^{-2}$ above the ground state energy is approximately equal to the spectrum of the Lieb-Liniger model (1.1) as long as $a \ll r$ and $r \ll \ell$. The effective coupling parameter g in (1.1) is of the order $g \sim a/r^2$ and can take any value in $[0, \infty]$. The same result applies to the corresponding eigenfunctions. They are approximately given by the corresponding eigenfunctions of the Lieb-Liniger Hamiltonian, multiplied by a product function of the variables orthogonal to z. The precise statement of our results will be given in the next section.

We remark that the problem considered here is somewhat analogous to the one-dimensional behavior of atoms in extremely strong magnetic fields, where the Coulomb interaction behaves like an effective one-dimensional δ-potential when the

magnetic field shrinks the cyclotron radius of the electrons to zero. For such systems, the asymptotics of the ground state energy was studied in [1], and later the excitation spectrum and corresponding eigenfunctions were investigated in [4]. In this case, the effective one-dimensional potential can be obtained formally by integrating out the variables transverse to the magnetic field in a suitably scaled Coulomb potential. Our case considered here is much more complicated, however. The correct one-dimensional physics emerges only if the kinetic and potential parts of the Hamiltonian are considered together.

2 Main Results

Consider the Hamiltonian for n spinless bosons in three space dimensions, interacting via a pair-potential v. We shall write $\mathbf{x} = (\mathbf{x}^\perp, z) \in \mathbb{R}^3$, with $\mathbf{x}^\perp \in \mathbb{R}^2$. Let $V^\perp \in L^\infty_{\rm loc}(\mathbb{R}^2)$ and $V^\parallel \in L^\infty_{\rm loc}(\mathbb{R})$ denote the (real-valued) confining potentials in $\mathbf{x}^\perp$ in z-direction, respectively. Then

$$H_{\rm 3d}^{n,r,\ell,a} = \sum_{i=1}^n \Big(-\Delta_i + r^{-2} V^\perp(\mathbf{x}_i^\perp/r) + \ell^{-2} V^\parallel(z_i/\ell) \Big) + \sum_{1\leq i<j\leq n} a^{-2} v(|\mathbf{x}_i - \mathbf{x}_j|/a)\,. \tag{2.1}$$

The trap potentials $V^\parallel$ and $V^\perp$ confine the motion in the longitudinal (z) and the transversal ($\mathbf{x}^\perp$) directions, respectively, and are assumed to be locally bounded and tend to ∞ as $|z|$ and $|\mathbf{x}^\perp|$ tend to ∞. Without loss of generality, we can assume that $V^\parallel \geq 0$. The scaling parameters r and ℓ measure the size of the traps.

The interaction potential v is assumed to be nonnegative, of finite range R_0 and have scattering length 1; the scaled potential $a^{-2}v(|\cdot|/a)$ then has scattering length a [14, 15] and range aR_0. We do not assume any smoothness or even integrability of v. In particular, we allow v to take the value $+\infty$ on a set of non-zero measure, corresponding to hard-sphere particles.

Let $e^\perp$ and $b(\mathbf{x}^\perp)$ denote the ground state energy and the normalized ground state wave function of $-\Delta^\perp + V^\perp(\mathbf{x}^\perp)$, respectively. Note that b is a bounded and strictly positive function and, in particular, $b \in L^p(\mathbb{R}^2)$ for any $2 \leq p \leq \infty$. Let also $\widetilde{e}^\perp > 0$ denote the gap above the ground state energy of $-\Delta^\perp + V^\perp(\mathbf{x}^\perp)$. The corresponding quantities for $-\Delta^\perp + r^{-2}V^\perp(\mathbf{x}^\perp/r)$ are then given by $e^\perp/r^2$, $\widetilde{e}^\perp/r^2$ and $b_r(\mathbf{x}^\perp) = r^{-1}b(\mathbf{x}^\perp/r)$, respectively.

The eigenvalues of $H_{\rm 3d}^{n,r,\ell,a}$ will be denoted by $E^k_{\rm 3d}(n,r,\ell,a)$, with $k = 1,2,3,\ldots$. Moreover, the eigenvalues of $H_{\rm 1d}^{n,\ell,g}$ in (1.1) will be denoted by $E^k_{\rm 1d}(n,\ell,g)$. Theorem 1 shows that $E^k_{\rm 3d}(n,r,\ell,a)$ is approximately equal to $E^k_{\rm 1d}(n,\ell,g) + e^\perp/r^2$ for small a/r and r/ℓ, for an appropriate value of the parameter g. In fact, g turns out to be given by

$$g = \frac{8\pi a}{r^2} \int_{\mathbb{R}^2} |b(\mathbf{x}^\perp)|^4 d^2\mathbf{x}^\perp\,. \tag{2.2}$$

THEOREM 1. *Let g be given in (2.2). There exist constants $C > 0$ and $D > 0$, independent of a, r, ℓ, n and k, such that the following bounds hold:*

$$\text{(a)} \quad E^k_{\rm 3d}(n,r,\ell,a) \geq \frac{ne^\perp}{r^2} + E^k_{\rm 1d}(n,\ell,g)\,(1-\eta_{\rm L})\left(1 - \frac{r^2}{\widetilde{e}^\perp}E^k_{\rm 1d}(n,\ell,g)\right) \tag{2.3}$$

as long as $E^k_{\rm 1d}(n,\ell,g) \leq \widetilde{e}^\perp/r^2$, with

$$\eta_{\rm L} = D\left[\left(\frac{na}{r}\right)^{1/8} + n^2\left(\frac{na}{r}\right)^{3/8}\right] \tag{2.4}$$

$$\text{(b)} \qquad E^k_{\rm 3d}(n,r,\ell,a) \leq \frac{ne^\perp}{r^2} + E^k_{\rm 1d}(n,\ell,g)\,(1-\eta_{\rm U})^{-1} \tag{2.5}$$

whenever $\eta_{\rm U} < 1$, where

$$\eta_{\rm U} = C\left(\frac{na}{r}\right)^{2/3} . \tag{2.6}$$

We note that $E^k_{\rm 1d}(n,\ell,g)$ is monotone increasing in g, and uniformly bounded in g for fixed k. In fact, $E^k_{\rm 1d}(n,\ell,g) \leq E^k_{\rm f}(n,\ell)$ for all $g \geq 0$ and for all k, where $E^k_{\rm f}(n,\ell)$ are the eigenenergies of n non-interacting fermions in one dimension [8]. Using this property of uniform boundedness, we can obtain the following Corollary from Theorem 1.

COROLLARY 1. *Fix k, n and ℓ. If $r \to 0$, $a \to 0$ in such a way that $a/r \to 0$, then*

$$\lim \frac{E^k_{\rm 3d}(n,r,\ell,a) - ne^\perp/r^2}{E^k_{\rm 1d}(n,\ell,g)} = 1 . \tag{2.7}$$

Note that by simple scaling $E^k_{\rm 3d}(n,r,\ell,a) = \ell^{-2}E^k_{\rm 3d}(n,r/\ell,1,a/\ell)$, and likewise $E^k_{\rm 1d}(n,\ell,g) = \ell^{-2}E^k_{\rm 1d}(n,1,\ell g)$. Hence it is no restriction to fix ℓ in the limit considered in Corollary 1. In fact, ℓ could be set equal to 1 without loss of generality. Note also that the convergence stated in Corollary 1 is uniform in g, in the sense that $g \sim ar^{-2}$ is allowed to go to $+\infty$ as $r \to 0$, $a \to 0$, as long as $gr \sim a/r \to 0$.

In order to state our results on the corresponding eigenfunctions of $H^{n,r,\ell,a}_{\rm 3d}$ and $H^{n,\ell,g}_{\rm 1d}$, we first have to introduce some additional notation to take into account the possible degeneracies of the eigenvalues. Let $k_1 = 1$, and let k_i be recursively defined by

$$k_i = \min\left\{k \,:\, E^k_{\rm 1d}(n,\ell,g) > E^{k_{i-1}}_{\rm 1d}(n,\ell,g)\right\} .$$

Then $E^{k_i}_{\rm 1d}(n,\ell,g) < E^{k_{i+1}}_{\rm 1d}(n,\ell,g)$, while $E^{k_i+j}_{\rm 1d}(n,\ell,g) = E^{k_i}_{\rm 1d}(n,\ell,g)$ for $0 \leq j < k_{i+1} - k_i$. That is, k counts the energy levels including multiplicities, while i counts the levels without multiplicities. Hence, if $k_i \leq k < k_{i+1}$, the energy eigenvalue $E^k_{\rm 1d}(n,\ell,g)$ is $k_{i+1} - k_i$ fold degenerate. Note that k_i depends on n, ℓ and g, of course, but we suppress this dependence in the notation for simplicity.

Our main result concerning the eigenfunctions of $H^{n,r,\ell,a}_{\rm 3d}$ is as follows.

THEOREM 2. *Let g be given in (2.2). Let Ψ_k be an eigenfunction of $H_{\rm 3d}^{n,r,\ell,a}$ with eigenvalue $E_{\rm 3d}^k(n,r,\ell,a)$, and let ψ_l ($k_i \leq l < k_{i+1}$) be orthonormal eigenfunctions of $H_{\rm 1d}^{n,\ell,g}$ corresponding to eigenvalue $E_{\rm 1d}^k(n,\ell,g)$. Then*

$$\sum_{l=k_i}^{k_{i+1}-1} \left| \left\langle \Psi_k \middle| \psi_l \prod_{i=1}^n b_r(\mathbf{x}_k^\perp) \right\rangle \right|^2 \geq 1 - \left(\frac{\big(1-(r^2/\widetilde{e}^\perp)E_{\rm 1d}^k(n,\ell,g)\big)^{-1}}{(1-\eta_{\rm U})(1-\eta_{\rm L})} - 1 \right) \times \\ \times \left[\frac{\sum_{i=1}^{k_{i+1}-1} E_{\rm 1d}^i(n,\ell,g)}{E_{\rm 1d}^{k_{i+1}}(n,\ell,g) - E_{\rm 1d}^{k_i}(n,\ell,g)} + \frac{\sum_{i=1}^{k_i-1} E_{\rm 1d}^i(n,\ell,g)}{E_{\rm 1d}^{k_i}(n,\ell,g) - E_{\rm 1d}^{k_i-1}(n,\ell,g)} \right] \tag{2.8}$$

as long as $\eta_{\rm U} < 1$, $\eta_{\rm L} < 1$ and $E_{\rm 1d}^k(n,\ell,g) < \widetilde{e}^\perp/r^2$.

In particular, this shows that $\Psi_k(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is approximately of the product form $\psi_k(z_1,\dots,z_n)\prod_i b_r(\mathbf{x}_i^\perp)$ for small r/ℓ and a/r, where ψ_k is an eigenfunction of $H_{\rm 1d}^{n,\ell,g}$ with eigenvalue $E_{\rm 1d}^k(n,\ell,g)$. Note that although Ψ_k is close to such a product in $L^2(\mathbb{R}^{3n})$ sense, it is certainly not close to a product in a stronger norm involving the energy. In fact, a product wavefunction will have infinite energy if the interaction potential v contains a hard core; in any case, its energy will be too big and not related to the scattering length of v at all.

COROLLARY 2. *Fix k, n, ℓ and $g_0 \geq 0$. Let $P_{g_0,{\rm 1d}}^k$ denote the projection onto the eigenspace of $H_{\rm 1d}^{n,\ell,g_0}$ with eigenvalue $E_{\rm 1d}^k(n,\ell,g_0)$, and let $P_r^\perp$ denote the projection onto the function $\prod_{i=1}^n b_r(\mathbf{x}_i^\perp) \in L^2(\mathbb{R}^{2n})$. If $r \to 0$, $a \to 0$ in such a way that $g = 8\pi\|b\|_4^4 a/r^2 \to g_0$, then*

$$\lim \langle \Psi_k | P_r^\perp \otimes P_{g_0,{\rm 1d}}^k | \Psi_k \rangle = 1\,. \tag{2.9}$$

Here, the tensor product refers to the decomposition $L^2(\mathbb{R}^{3n}) = L^2(\mathbb{R}^{2n}) \otimes L^2(\mathbb{R}^n)$ into the transversal ($\mathbf{x}^\perp$) and longitudinal (z) variables.

We note that Corollary 2 holds also in case $g_0 = \infty$. In this case, $P_{g_0,{\rm 1d}}^k$ has to be defined as the spectral projection with respect to the limiting energies $\lim_{g\to\infty} E_{\rm 1d}^k(n,\ell,g)$. Using compactness, it is in fact easy to see that the limit of $P_{g_0,{\rm 1d}}^k$ as $g_0 \to \infty$ exists in the operator norm topology. We omit the details.

Our results in particular imply that $H_{\rm 3d}^{n,r,\ell,a}$ converges to $H_{\rm 1d}^{n,\ell,g}$ in a certain norm-resolvent sense. In fact, if $a \to 0$ and $r \to 0$ in such a way that $g = 8\pi\|b\|_4^4 a/r^2 \to g_0$, and $\lambda \in \mathbb{C} \setminus [E_{\rm 1d}^1(n,\ell,g_0),\infty)$ is fixed, then it follows easily from Corollaries 1 and 2 that

$$\lim \left\| \frac{1}{\lambda + ne^\perp/r^2 - H_{\rm 3d}^{n,r,\ell,a}} - P_r^\perp \otimes \frac{1}{\lambda - H_{\rm 1d}^{n,\ell,g_0}} \right\| = 0\,.$$

Our main results, Theorems 1 and 2, can be extended in several ways, as we will explain now. For simplicity and transparency, we shall not formulate the proofs in the most general setting.

- Instead of allowing the whole of $\mathbb{R}^2$ as the configuration space for the $\mathbf{x}^\perp$ variables, one could restrict it to a subset, with appropriate boundary conditions for the Laplacian $\Delta^\perp$. For instance, if $V^\perp$ is zero and the motion is restricted to a disk with Dirichlet boundary conditions, the corresponding ground state function b in the transversal directions is given by a Bessel function.

- Similarly, instead of taking $\mathbb{R}$ as the configuration space for the z variables, one can work on an interval with appropriate boundary conditions. In particular, the case $V^\parallel = 0$ with periodic boundary conditions on $[0, \ell]$ can be considered, which is the special case studied by Lieb and Liniger in [12, 11].

- As noted in previous works on dilute Bose gases [15, 14], the restriction of v having a finite range can be dropped. Corollaries 1 and 2 remain true for all repulsive interaction potentials v with finite scattering length. Also Theorems 1 and 2 remain valid, with possibly modified error terms, however, depending on the rate of decay of v at infinity. We refer to [15, 14] for details.

- Our results can be extended to any symmetry type of the 1D wavefunctions, not just symmetric ones. In particular, one can allow the particles to have internal degrees of freedom, like spin.

- As mentioned in the Introduction, in the special case $k = 1$, i.e., for the ground state energy, similar bounds as in Theorem 1 have been obtained in [17]. In spite of the fact that the error terms are not uniform in the particle number n, these bounds have then been used to estimate the ground state energy in the thermodynamic limit, using the technique of Dirichlet–Neumann bracketing. Combining this technique with the results of Theorem 1, one can obtain appropriate bounds on the free energy at positive temperature and other thermodynamic potentials in the thermodynamic limit as well. In fact, since our energy bounds apply to all (low-lying) energy eigenvalues, bounds on the free energy in a finite volume are readily obtained. The technique employed in [17] then allows for an extension of these bounds to infinite volume (at fixed particle density).

In the following, we shall give the proof of Theorems 1 and 2. The next Section 3 gives the proof of the upper bound to the energies $E^k_{\rm 3d}(n, r, \ell, a)$, as stated in Theorem 1(b). The corresponding lower bounds in Theorem 1(a) are proved in Section 4. Finally, the proof of Theorem 2 will be given in Section 5.

3 Upper Bounds

This section contains the proof of the upper bounds to the 3D energies $E^k_{\rm 3d}(n, r, \ell, a)$, stated in Theorem 1(b). Our strategy is similar to the one in [17, Sect. 3.1] and we will use some of the estimates derived there. The main improvements presented here concern the extension to excited energy eigenvalues, and the derivation of a bound that is uniform in the effective coupling constant g in (2.2). In contrast, the upper

bound in [17, Thm. 3.1] applies only to the ground state energy $E^1_{3\mathrm{d}}(n,r,\ell,a)$, and is not uniform in g for large g.

Before proving the upper bounds to the 3D energies $E^k_{3\mathrm{d}}(n,r,\ell,a)$, we will prove a simple lemma that will turn out to be useful in the following.

LEMMA 1. *Let H be a non-negative Hamiltonian on a Hilbertspace $\mathcal{H}$, with eigenvalues $0 \leq E_1 \leq E_2 \leq E_3 \leq \dots$. For $k \geq 1$, let $f_1,\dots,f_k \in \mathcal{H}$. If, for any $\{a_i\}$ with $\sum_{i=1}^k |a_i|^2 = 1$ we have*

$$\Big\| \textstyle\sum_{i=1}^k a_i f_i \Big\|^2 \geq 1-\varepsilon \quad \textit{for } \varepsilon < 1$$

and

$$\Big\langle \textstyle\sum_{i=1}^k a_i f_i \Big| H \Big| \textstyle\sum_{i=1}^k a_i f_i \Big\rangle \leq E\,,$$

then $E_k \leq E(1-\varepsilon)^{-1}$.

Proof. For any f in the k-dimensional subspace spanned by the f_i, we have

$$\langle f|H|f\rangle \leq E(1-\varepsilon)^{-1}\langle f|f\rangle$$

by assumption. Hence $E_k \leq E/(1-\varepsilon)$ by the variational principle. ∎

Pick $\{a_i\}$ with $\sum_{i=1}^k |a_i|^2 = 1$, and let $\phi = \sum_{i=1}^k a_i\psi_i$, where the ψ_i are orthonormal eigenfunctions of $H^{n,\ell,g}_{1\mathrm{d}}$ with eigenvalues $E^i_{1\mathrm{d}}(n,\ell,g)$. Consider a 3D trial wave function of the form

$$\Phi(\mathbf{x}_1,\dots,\mathbf{x}_n) = \phi(z_1,\dots,z_n)F(\mathbf{x}_1,\dots,\mathbf{x}_n)\prod_{k=1}^n b_r(\mathbf{x}_k^\perp)\,,$$

where F is defined by

$$F(\mathbf{x}_1,\dots,\mathbf{x}_n) = \prod_{i<j} f(|\mathbf{x}_i - \mathbf{x}_j|)\,.$$

Here, f is a function with $0 \leq f \leq 1$, monotone increasing, such that $f(t) = 1$ for $t \geq R$ for some $R \geq aR_0$. For $t \leq R$ we shall choose $f(t) = f_0(t)/f_0(R)$, where f_0 is the solution to the zero-energy scattering equation for $a^{-2}v(|\cdot|/a)$ [14, 15]. That is,

$$\left(-\frac{d^2}{dt^2} - \frac{2}{t}\frac{d}{dt} + \frac{1}{2a^2}v(t/a)\right) f_0(t) = 0\,,$$

normalized such that $\lim_{t\to\infty} f_0(t) = 1$. This f_0 has the properties that $f_0(t) = 1-a/t$ for $t \geq aR_0$, and $f_0'(t) \leq t^{-1}\min\{1, a/t\}$.

To be able to apply Lemma 1, we need a lower bound on the norm of Φ. This can be obtained in the same way as in [17, Eq. (3.9)]. Let $\varrho^{(2)}_\phi$ denote the two-particle density of ϕ, normalized as

$$\int_{\mathbb{R}^2} \varrho^{(2)}_\phi(z,z')dzdz' = 1\,.$$

Since F is 1 if no pair of particles is closer together than a distance R, we can estimate the norm of Φ by

$$\begin{aligned}\langle\Phi|\Phi\rangle &\geq 1-\frac{n(n-1)}{2}\int_{\mathbb{R}^6}\varrho^{(2)}_\phi(z,z')b_r(\mathbf{x}^\perp)^2 b_r(\mathbf{y}^\perp)^2\theta(R-|\mathbf{x}-\mathbf{y}|)dzdz'd^2\mathbf{x}^\perp d^2\mathbf{y}^\perp \\ &\geq 1-\frac{n(n-1)}{2}\int_{\mathbb{R}^4} b_r(\mathbf{x}^\perp)^2 b_r(\mathbf{y}^\perp)^2\theta(R-|\mathbf{x}^\perp-\mathbf{y}^\perp|)d^2\mathbf{x}^\perp d^2\mathbf{y}^\perp \\ &\geq 1-\frac{n(n-1)}{2}\frac{\pi R^2}{r^2}\|b\|_4^4\,, \end{aligned} \tag{3.1}$$

where Young's inequality [13, Thm. 4.2] has been used in the last step. Since $F\leq 1$, the norm of Φ is less than 1, and hence

$$1\geq\langle\Phi|\Phi\rangle\geq 1-\frac{n(n-1)}{2}\frac{\pi R^2}{r^2}\|b\|_4^4\,. \tag{3.2}$$

Next, we will derive an upper bound on $\langle\Phi|H^{n,r,\ell,a}_{\rm 3d}|\Phi\rangle$. Define G by $\Phi=GF$. Using partial integration and the fact that F is real-valued, we have

$$\langle\Phi|-\Delta_j|\Phi\rangle=-\int_{\mathbb{R}^{3n}}F^2\overline{G}\Delta_j G+\int_{\mathbb{R}^{3n}}|G|^2|\nabla_j F|^2\,.$$

Using $-\Delta^\perp b_r=(e^\perp/r^2)b_r-r^{-2}V^\perp(\,\cdot\,/r)b_r$, we therefore get

$$\begin{aligned}\langle\Phi|H^{n,r,\ell,a}_{\rm 3d}|\Phi\rangle=&\ \frac{ne^\perp}{r^2}\langle\Phi|\Phi\rangle+\int_{\mathbb{R}^{3n}}F^2\prod_{j=1}^n b_r(\mathbf{x}_j^\perp)^2\,\overline{\phi}\left(H^{n,\ell,g}_{\rm 1d}\phi\right) \\ &-g\left\langle\Phi\left|\textstyle\sum_{i<j}\delta(z_i-z_j)\right|\Phi\right\rangle \\ &+\int_{\mathbb{R}^{3n}}|G|^2\left(\sum_{j=1}^n|\nabla_j F|^2+\sum_{i<j}a^{-2}v(|\mathbf{x}_i-\mathbf{x}_j|/a)|F|^2\right)\,. \end{aligned} \tag{3.3}$$

With the aid of Schwarz's inequality for the integration over the z variables, as well as $F\leq 1$,

$$\begin{aligned}&\int_{\mathbb{R}^{3n}}F^2\prod_{j=1}^n b_r(\mathbf{x}_j^\perp)^2\,\overline{\phi}\big(H^{n,\ell,g}_{\rm 1d}\phi\big) \\ &\leq\left(\int_{\mathbb{R}^{3n}}F^2\prod_{j=1}^n b_r(\mathbf{x}_j^\perp)^2|\phi|^2\right)^{1/2}\left(\int_{\mathbb{R}^{3n}}F^2\prod_{j=1}^n b_r(\mathbf{x}^\perp)^2\big|H^{n,\ell,g}_{\rm 1d}\phi\big|^2\right)^{1/2} \\ &\leq\|\phi\|_2\|H^{n,\ell,g}_{\rm 1d}\phi\|_2\leq E^k_{\rm 1d}(n,\ell,g)\,. \end{aligned} \tag{3.4}$$

Here, we have used that ϕ is a linear combination of the first k eigenfunctions of $H^{n,\ell,g}_{\rm 1d}$ to obtain the final inequality. The term in the second line of (3.3) is bounded by

$$\left\langle\Phi\left|\textstyle\sum_{i<j}\delta(z_i-z_j)\right|\Phi\right\rangle\geq\frac{n(n-1)}{2}\int_{\mathbb{R}}\varrho^{(2)}_\phi(z,z)dz\left(1-\frac{n(n-1)}{2}\frac{\pi R^2}{r^2}\|b\|_4^4\right)\,, \tag{3.5}$$

as an argument similar to (3.1) shows.

The remaining last term in (3.3) can be bounded in the similar way as in [17, Sect. 3]. In fact, by repeating the analysis in [17, Eqs. (3.12)–(3.19)]

$$
\begin{aligned}
&\int_{\mathbb{R}^{3n}} |G|^2 \left(\sum_{j=1}^{n} |\nabla_j F|^2 + \sum_{i<j} a^{-2} v(|\mathbf{x}_i - \mathbf{x}_j|/a)|F|^2 \right) \\
&\leq \frac{n(n-1)}{r^2} \|b\|_4^4 \int_{\mathbb{R}^2} \varrho_\phi^{(2)}(z,z') h(z-z') dz dz' \\
&\quad + \frac{2}{3} n(n-1)(n-2) \frac{\|b\|_\infty^2}{r^2} \frac{\|b\|_4^4}{r^2} \|m\|_\infty \int_{\mathbb{R}^2} \varrho_\phi^{(2)}(z,z') m(z-z') dz dz' \,.
\end{aligned} \tag{3.6}
$$

Here,

$$
h(z) = \int_{\mathbb{R}^2} \left(f'(|\mathbf{x}|)^2 + \tfrac{1}{2} a^{-2} v(|\mathbf{x}|/a) f(|\mathbf{x}|)^2 \right) d^2\mathbf{x}^\perp
$$

and

$$
m(z) = \int_{\mathbb{R}^2} f'(|\mathbf{x}|) d^2\mathbf{x}^\perp \,.
$$

(The function m was called k in [17].) Note that both h and m are supported in $[-R,R]$, and $\int_{\mathbb{R}} h(z)dz = 4\pi a(1-a/R)^{-1}$.

We now proceed differently than in [17]. Our analysis here has the advantage of yielding an upper bound that is uniform in g. Let $\varphi \in H^1(\mathbb{R})$. Then

$$
\begin{aligned}
\big| |\varphi(z)|^2 - |\varphi(z')|^2 \big| &= \left| \int_{z'}^{z} \frac{d|\varphi(t)|^2}{dt} dt \right| \\
&\leq 2|\varphi(z')| \int_{z'}^{z} \left| \frac{d\varphi(t)}{dt} \right| dt + 2 \left(\int_{z'}^{z} \left| \frac{d\varphi(t)}{dt} \right| dt \right)^2 \\
&\leq 2|\varphi(z')| |z-z'|^{1/2} \left(\int_{\mathbb{R}} \left| \frac{d\varphi}{dz} \right|^2 \right)^{1/2} + 2|z-z'| \left(\int_{\mathbb{R}} \left| \frac{d\varphi}{dz} \right|^2 \right) .
\end{aligned} \tag{3.7}
$$

Here, we used $|\varphi(t)| \leq |\varphi(z')| + \int_{z'}^{z} \left| \frac{d\varphi(t)}{dt} \right| dt$ for $z' \leq t \leq z$ for the first inequality and applied Schwarz's inequality for the second.

We apply the bound (3.7) to $\varrho_\phi^{(2)}(z,z')$ for fixed z'. Using the fact that the support of h is contained in $[-R,R]$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \varrho_\phi^{(2)}(z,z') h(z-z') dz dz' - \int_{\mathbb{R}} h(z) dz \int_{\mathbb{R}} \varrho_\phi^{(2)}(z,z) dz \\
&\leq 2R^{1/2} \int_{\mathbb{R}} h(z) dz \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \partial_z \sqrt{\varrho_\phi^{(2)}(z,z')} \right|^2 dz \right)^{1/2} \sqrt{\varrho_\phi^{(2)}(z',z')}\, dz' \\
&\quad + 2R \int_{\mathbb{R}} h(z) dz \int_{\mathbb{R}^2} \left| \partial_z \sqrt{\varrho_\phi^{(2)}(z,z')} \right|^2 dz dz' \\
&\leq \frac{2}{n} E_{\mathrm{1d}}^{k}(n,\ell,g) \int_{\mathbb{R}} h(z) dz \left[\left(\frac{2R}{(n-1)g} \right)^{1/2} + R \right] .
\end{aligned}
$$

Here, we used Schwarz's inequality and the fact that $g\frac{1}{2}n(n-1)\int_{\mathbb{R}}\varrho_\phi^{(2)}(z,z)dz \leq E_{1\mathrm{d}}^k(n,\ell,g)$ and

$$\int_{\mathbb{R}^2}\left|\partial_z\sqrt{\varrho_\phi^{(2)}(z,z')}\right|^2 dzdz' \leq \left\langle\phi\left|-\frac{d^2}{dz_1^2}\right|\phi\right\rangle \leq E_{1\mathrm{d}}^k(n,\ell,g)/n\,.$$

The same argument is used with h replaced by m. Now $\int_{\mathbb{R}} h(z)dz = 4\pi a(1-a/R)^{-1}$, and [17, Eq. (3.22)]

$$\begin{aligned}\|m\|_\infty &\leq \frac{2\pi a}{1-a/R}\left(1+\ln(R/a)\right)\,,\\ \int_{\mathbb{R}} m(z)dz &\leq \frac{2\pi aR}{1-a/R}\left(1-\frac{a}{2R}\right)\,.\end{aligned}$$

Therefore

$$(3.6) \leq \frac{n(n-1)}{2}g\frac{1+K}{1-a/R}\left(\int_{\mathbb{R}}\varrho_\phi^{(2)}(z,z)dz + \frac{2}{n}\left[\left(\frac{2R}{(n-1)g}\right)^{1/2}+R\right]E_{1\mathrm{d}}^k(n,\ell,g)\right) \tag{3.8}$$

where we denoted

$$K = \frac{2\pi}{3}(n-2)\frac{a}{R}\frac{1+\ln(R/a)}{1-a/R}\left(\frac{R}{r}\right)^2\|b\|_\infty^2\,.$$

Putting together the bounds (3.4), (3.5) and (3.8), and using again the fact that $g\frac{1}{2}n(n-1)\int_{\mathbb{R}}\varrho_\phi^{(2)}(z,z)dz \leq E_{1\mathrm{d}}^k(n,\ell,g)$, we obtain the upper bound

$$\begin{aligned}\left\langle\Phi\left|H_{3\mathrm{d}}^{n,r,\ell,a}-\frac{ne^\perp}{r^2}\right|\Phi\right\rangle \leq E_{1\mathrm{d}}^k(n,\ell,g)\Bigg(&1+\frac{n(n-1)}{2}\frac{\pi R^2}{r^2}\|b\|_4^4+\frac{a/R+K}{(1-a/R)}\\ &+\frac{1+K}{1-a/R}\left[(2Rg(n-1))^{1/2}+(n-1)Rg\right]\Bigg)\,.\end{aligned}$$

It remains to choose R. If we choose

$$R^3 = \frac{ar^2}{n^2}$$

then

$$\left\langle\Phi\left|H_{3\mathrm{d}}^{n,r,\ell,a}-\frac{ne^\perp}{r^2}\right|\Phi\right\rangle \leq E_{1\mathrm{d}}^k(n,\ell,g)\left(1+C\left(\frac{na}{r}\right)^{2/3}\right)$$

for some constant $C>0$. Moreover, from (3.2) we see that

$$\langle\Phi|\Phi\rangle \geq 1-C\left(\frac{na}{r}\right)^{2/3}\,.$$

Hence the upper bound (2.5) of Theorem 1 follows with the aid of Lemma 1.

4 Lower Bounds

In this section, we will derive a lower bound on the operator $H_{3\mathrm{d}}^{n,r,\ell,a}$ in terms of the Lieb-Liniger Hamiltonian $H_{1\mathrm{d}}^{n,\ell,g}$. In particular, this will prove the desired lower bounds on the energies $E_{3\mathrm{d}}^{k}(n,r,\ell,a)$. Our method is based on [17, Thm. 3.1], but extends it in several important ways which we shall explain.

Let Φ be a normalized wavefunction in $L^2(\mathbb{R}^{3n})$. We define $f \in L^2(\mathbb{R}^n)$ by

$$f(z_1,\ldots,z_n) = \int_{\mathbb{R}^{2n}} \Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n) \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp) d^2\mathbf{x}_k^\perp \, . \tag{4.1}$$

Moreover, we define F by

$$\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n) = F(\mathbf{x}_1,\ldots,\mathbf{x}_n) \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp) \, .$$

Note that F is well-defined, since b_r is a strictly positive function. Finally, let G be given by

$$G(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n) - f(z_1,\ldots,z_n) \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp) \, . \tag{4.2}$$

Using partial integration and the eigenvalue equation for b_r, we obtain

$$\begin{aligned} \left\langle \Phi \left| H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^\perp}{r^2} \right| \Phi \right\rangle = \sum_{i=1}^{n} \int_{\mathbb{R}^{3n}} \Bigg[& |\nabla_i F|^2 + \ell^{-2} V^{\|}(z_i/\ell) |F|^2 \\ & + \tfrac{1}{2} \sum_{j,\, j\neq i} a^{-2} v(|\mathbf{x}_i - \mathbf{x}_j|/a) |F|^2 \Bigg] \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp)^2 d^3\mathbf{x}_k. \end{aligned} \tag{4.3}$$

Now choose some $R > aR_0$, and let

$$U(r) = \begin{cases} 3(R^3 - aR_0^3)^{-1} & \text{for } aR_0 \leq r \leq R \\ 0 & \text{otherwise .} \end{cases}$$

For $\delta > 0$ define $\mathcal{B}_\delta \subset \mathbb{R}^2$ by

$$\mathcal{B}_\delta = \left\{ \mathbf{x}^\perp \in \mathbb{R}^2 \, : \, b(\mathbf{x}^\perp)^2 \geq \delta \right\} \, .$$

Proceeding along the same lines as in [17, Eqs. (3.31)–(3.36)], we conclude that, for any $0 \leq \varepsilon \leq 1$,

$$\begin{aligned} & \sum_{i=1}^{n} \int_{\mathbb{R}^{3n}} \left[|\nabla_i F|^2 + \tfrac{1}{2} \sum_{j,\, j\neq i} a^{-2} v(|\mathbf{x}_i - \mathbf{x}_j|/a) |F|^2 \right] \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp)^2 d^3\mathbf{x}_k \\ & \geq \sum_{i=1}^{n} \int_{\mathbb{R}^{3n}} \left[\varepsilon |\nabla_i^\perp F|^2 + a' U(|\mathbf{x}_i - \mathbf{x}_{k(i)}|) \chi_{\mathcal{B}_\delta}(\mathbf{x}_{k(i)}^\perp / r) |F|^2 \right] \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp)^2 d^3\mathbf{x}_k \\ & \quad + \int_{\mathbb{R}^{3n}} \left[\varepsilon |\partial_i \Phi|^2 + (1-\varepsilon) |\partial_i \Phi|^2 \chi_{\min_k |\mathbf{x}_i - \mathbf{x}_k| \geq R}(\mathbf{x}_i) \right] \prod_{k=1}^{n} d^3\mathbf{x}_k \, . \end{aligned} \tag{4.4}$$

Here, $\mathbf{x}_{k(i)}$ denote the nearest neighbor of $\mathbf{x}_i$ among the $\mathbf{x}_j$ with $j \neq i$, $\mathbf{x}_{k(i)}^\perp$ is its $\perp$-component, and $a' = a(1-\varepsilon)(1 - 2R\|\nabla b^2\|_\infty/r\delta)$. (It is easy to see that ∇b^2 is a bounded function; see, e.g., the proof of Lemma 1 in the Appendix in [17].) The characteristic function of $\mathcal{B}_\delta$ is denoted by $\chi_{\mathcal{B}_\delta}$, and $\chi_{\min_k |\mathbf{x}_i - \mathbf{x}_k| \geq R}$ restricts the $\mathbf{x}_i$ integration to the complement of the balls of radius R centered at the $\mathbf{x}_k$ for $k \neq i$. That is, for the lower bound in (4.4) only the kinetic energy inside these balls gets used. (Compared with [17, Eq. (3.36)], we have dropped part of the kinetic energy in the $\mathbf{x}^\perp$ direction in the last line, which is legitimate for a lower bound.)

For a lower bound, the characteristic function $\chi_{\min_k |\mathbf{x}_i - \mathbf{x}_k| \geq R}$ could be replaced by the smaller quantity $\chi_{\min_k |z_i - z_k| \geq R}$, as was done in [17]. We do not do this here, however, and this point will be important in the following. In particular, it allows us to have the full kinetic energy (in the z direction) at our disposal in the effective one-dimensional problem that is obtained after integrating out the $\mathbf{x}^\perp$ variables. In contrast, only the kinetic energy in the regions $|z_i - z_k| \geq R$ was used in [17] to derive a lower bound on the ground state energy. The improved method presented in the following leads to an operator lower bound, however.

We now give a lower bound on the two terms on the right side of (4.4). We start with the second term, which is bounded from below by

$$\begin{aligned}
&\int_{\mathbb{R}^{3n}} |\partial_i \Phi|^2 \prod_{k=1}^n d^3\mathbf{x}_k \\
&- (1-\varepsilon) \int_{\mathbb{R}^{3n}} |\partial_i \Phi|^2 \chi_{\min_k |z_i - z_k| \leq R}(z_i) \chi_{\min_k |\mathbf{x}_i^\perp - \mathbf{x}_k^\perp| \leq R}(\mathbf{x}_i^\perp) \prod_{k=1}^n d^3\mathbf{x}_k \,.
\end{aligned}$$

To estimate the last term from below, consider first the integral over the $\mathbf{x}^\perp$ variables for fixed values of $z_1, \dots, z_n$. Using (4.2), this integral equals

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \left| \partial_i f \textstyle\prod_{k=1}^n b_r(\mathbf{x}_k^\perp) + \partial_i G \right|^2 \chi_{\min_k |\mathbf{x}_i^\perp - \mathbf{x}_k^\perp| \leq R}(\mathbf{x}_i^\perp) \displaystyle\prod_{k=1}^n d^2\mathbf{x}_k^\perp \\
&\leq \left(1 + \eta^{-1}\right) |\partial_i f|^2 \int_{\mathbb{R}^{2n}} \chi_{\min_k |\mathbf{x}_i^\perp - \mathbf{x}_k^\perp| \leq R}(\mathbf{x}_i^\perp) \prod_{k=1}^n b_r(\mathbf{x}_k^\perp)^2 d^2\mathbf{x}_k^\perp \\
&\quad + (1+\eta) \int_{\mathbb{R}^{2n}} |\partial_i G|^2 \prod_{k=1}^n d^2\mathbf{x}_k^\perp
\end{aligned}$$

for any $\eta > 0$, by Schwarz's inequality. It is easy to see that

$$\int_{\mathbb{R}^{2n}} \chi_{\min_k |\mathbf{x}_i^\perp - \mathbf{x}_k^\perp| \leq R}(\mathbf{x}_i^\perp) \prod_{k=1}^n b_r(\mathbf{x}_k^\perp)^2 d^2\mathbf{x}_k^\perp \leq \frac{n(n-1)}{2} \frac{\pi R^2}{r^2} \|b\|_4^4$$

using Young's inequality, as in (3.1).

Now

$$\int_{\mathbb{R}^{3n}} |\partial_i \Phi|^2 \prod_{k=1}^n d^3\mathbf{x}_k = \int_{\mathbb{R}^{3n}} |\partial_i f|^2 \prod_{k=1}^n b_r(\mathbf{x}_k^\perp)^2 d^3\mathbf{x}_k + \int_{\mathbb{R}^{3n}} |\partial_i G|^2 \prod_{k=1}^n d^3\mathbf{x}_k \,,$$

since, by the definition of G in (4.2), $\partial_i G$ is orthogonal to any function of the form $\xi(z_1, \ldots, z_n) \prod_k b_r(\mathbf{x}_k^\perp)$. We have thus shown the the second term on the right side of (4.4) satisfies the lower bound

$$\int_{\mathbb{R}^{3n}} \left[\varepsilon |\partial_i \Phi|^2 + (1-\varepsilon) |\partial_i \Phi|^2 \chi_{\min_k |\mathbf{x}_i - \mathbf{x}_k| \geq R}(\mathbf{x}_i) \right] \prod_{k=1}^n d^3\mathbf{x}_k$$
$$\geq \left(1 - (1-\varepsilon)\left(1+\eta^{-1}\right) \frac{n(n-1)}{2} \frac{\pi R^2}{r^2} \|b\|_4^4 \right) \int_{\mathbb{R}^n} |\partial_i f|^2 \prod_{k=1}^n dz_k$$
$$+ (1 - (1-\varepsilon)(1+\eta)) \int_{\mathbb{R}^{3n}} |\partial_i G|^2 \prod_{k=1}^n d^3\mathbf{x}_k \tag{4.5}$$

for any $\eta > 0$. We are going to choose η small enough such that $(1-\varepsilon)(1+\eta) \leq 1$, in which case the last term is non-negative and can be dropped for a lower bound. Note that the right side of (4.5) contains the full kinetic energy of the function f. Had we replaced $\chi_{\min_k |\mathbf{x}_i - \mathbf{x}_k| \geq R}$ by the smaller quantity $\chi_{\min_k |z_i - z_k| \geq R}$ on the left side of (4.5), as in [17], only the kinetic energy for $|z_i - z_k| \geq R$ would be at our disposal. This is sufficient for a lower bound on the ground state energy, as in [17], but would not lead to the desired operator lower bound which is derived in this section.

We now investigate the first term on the right side of (4.4). Consider, for fixed $z_1, \ldots, z_n$, the expression

$$\sum_{i=1}^n \int_{\mathbb{R}^{2n}} \left[\varepsilon |\nabla_i^\perp F|^2 + a' U(|\mathbf{x}_i - \mathbf{x}_{k(i)}|) \chi_{\mathcal{B}_\delta}(\mathbf{x}_{k(i)}^\perp / r) |F|^2 \right] \prod_{k=1}^n b_r(\mathbf{x}_k^\perp)^2 d^2\mathbf{x}_k^\perp \ . \tag{4.6}$$

Proceeding as in [17, Eqs. (3.39)–(3.46)], we conclude that

$$(4.6) \geq a'' \sum_{i<j} d(z_i - z_j) \int_{\mathbb{R}^{2n}} |\Phi(\mathbf{x}_i, \ldots, \mathbf{x}_n)|^2 \prod_{k=1}^n d^2\mathbf{x}_k^\perp \ ,$$

where

$$a'' = a' \left(1 - \frac{3n}{\varepsilon \widetilde{e}^\perp} \frac{ar^2}{R^3} \frac{1}{1-(aR_0/R)^3} \left[1 - \frac{n^2}{\varepsilon \widetilde{e}^\perp} \frac{a}{R} 3\pi \|b\|_4^4 \frac{1}{1-(aR_0/R)^3} \right]^{-1} \right)$$
$$\times \left(1 - (n-2) \frac{\pi R^2}{r^2} \|b\|_\infty^2 \right) \tag{4.7}$$

(which is called a''' in [17]), and

$$d(z - z') = \int_{\mathbb{R}^4} b_r(\mathbf{x}^\perp)^2 b_r(\mathbf{y}^\perp)^2 U(|\mathbf{x} - \mathbf{y}|) \chi_{\mathcal{B}_\delta}(\mathbf{y}^\perp / r) d^2\mathbf{x}^\perp d^2\mathbf{y}^\perp \ .$$

Note that $d(z) = 0$ if $|z| \geq R$. Recall that the quantity $\widetilde{e}^\perp$ in (4.7) is the gap above the ground state energy of $-\Delta^\perp + V^\perp$, which is strictly positive.

By using (4.1)–(4.2),

$$\int_{\mathbb{R}^{2n}} |\Phi(\mathbf{x}_i, \ldots, \mathbf{x}_n)|^2 \prod_{k=1}^{n} d^2\mathbf{x}_k^\perp \geq |f(z_1, \ldots, z_n)|^2 \,.$$

Let $g' = a'' \int_{\mathbb{R}} d(z)dz$. It was shown in [17, Eqs. (3.49)–(3.53)] that, for any $\varkappa > 0$ and $\varphi \in H^1(\mathbb{R})$,

$$\begin{aligned} & a'' \int_{\mathbb{R}} d(z - z')|\varphi(z)|^2 dz \\ & \geq g' \max_{z,\, |z-z'| \leq R} |\varphi(z)|^2 \left(1 - \sqrt{\frac{2g'R}{\varkappa}}\right) - \varkappa \int_{\mathbb{R}} |\partial\varphi|^2 dz \end{aligned}$$

for fixed z'. In particular, applying this estimate to f for fixed z_j, $j \neq i$,

$$\begin{aligned} & a'' \sum_{i<j} \int_{\mathbb{R}^n} d(z_i - z_j)|f|^2 \prod_{k=1}^{n} dz_k \\ & \geq g'' \sum_{i<j} \int_{\mathbb{R}^n} \delta(z_i - z_j)|f|^2 \prod_{k=1}^{n} dz_k - (n-1)\varkappa \sum_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_i f|^2 \prod_{k=1}^{n} dz_k \,, \end{aligned}$$

with

$$g'' = g' \left(1 - \sqrt{\frac{2g'R}{\varkappa}}\right) \,. \tag{4.8}$$

Hence we have shown that

$$\begin{aligned} & \sum_{i=1}^{n} \int \left[\varepsilon|\nabla_i^\perp F|^2 + a'U(|\mathbf{x}_i - \mathbf{x}_{k(i)}|)\chi_{\mathcal{B}_\delta}(\mathbf{x}_{k(i)}^\perp/r)|F|^2\right] \prod_{k=1}^{n} b_r(\mathbf{x}_k^\perp)^2 d^3\mathbf{x}_k \\ & \geq g'' \sum_{i<j} \int_{\mathbb{R}^n} \delta(z_i - z_j)|f|^2 \prod_{k=1}^{n} dz_k - (n-1)\varkappa \sum_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_i f|^2 \prod_{k=1}^{n} dz_k \,. \end{aligned} \tag{4.9}$$

This concludes our bound on the first term on the right side of (4.4).

For the term involving $V^\parallel$ in (4.3), we can again use the orthogonality properties of G in (4.2), as well as the assumed positivity of $V^\parallel$, to conclude that

$$\begin{aligned} & \sum_{i=1}^{n} \int_{\mathbb{R}^{3n}} \ell^{-2}V^\parallel(z_i/\ell)|\Phi(\mathbf{x}_1, \ldots, \mathbf{x}_n)|^2 \prod_{k=1}^{n} d^3\mathbf{x}_k \\ & \geq \sum_{i=1}^{n} \int_{\mathbb{R}^n} \ell^{-2}V^\parallel(z_i/\ell)|f(z_1, \ldots, z_n)|^2 \prod_{k=1}^{n} dz_k \,. \end{aligned} \tag{4.10}$$

By combining (4.3) and (4.4) with the estimates (4.5), (4.9) and (4.10), we thus

obtain

$$
\begin{aligned}
&\left\langle \Phi \left| H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^{\perp}}{r^2} \right| \Phi \right\rangle \\
&\geq \left(1 - (1-\varepsilon)\left(1+\eta^{-1}\right) \frac{n(n-1)}{2} \frac{\pi R^2}{r^2} \|b\|_4^4 - (n-1)\varkappa \right) \int_{\mathbb{R}^n} |\partial_i f|^2 \prod_{k=1}^n dz_k \\
&\quad + \int_{\mathbb{R}^n} \left[\sum_{i=1}^n \ell^{-2} V^{\parallel}(z_i/\ell) + g'' \sum_{i<j} \delta(z_i - z_j) \right] |f|^2 \prod_{k=1}^n dz_k \,. \qquad (4.11)
\end{aligned}
$$

Recall that g'' is given in (4.8). This estimate is valid for all $0 \leq \varepsilon \leq 1$, $\varkappa > 0$ and $\eta > 0$ such that $(1-\varepsilon)(1+\eta) \leq 1$ in order to be able to drop the last term in (4.5).

To complete the estimate, it remains to give a lower bound to $\int_{\mathbb{R}} d(z)dz$. As in [17, Eq. (3.48)], we can use $|b(\mathbf{x}^{\perp})^2 - b(\mathbf{y}^{\perp})^2| \leq R\|\nabla b^2\|_\infty$ for $|\mathbf{x}^{\perp} - \mathbf{y}^{\perp}| \leq R$ to estimate

$$
\begin{aligned}
\int_{\mathbb{R}} d(z)dz &\geq \frac{4\pi}{r^2} \left(\int_{\mathcal{B}_\delta} b(\mathbf{x}^{\perp})^4 d^2\mathbf{x}^{\perp} - R\|\nabla b^2\|_\infty / r \right) \\
&\geq \frac{4\pi}{r^2} \left(\|b\|_4^4 - \delta - R\|\nabla b^2\|_\infty / r \right) .
\end{aligned}
$$

This leads to the bound

$$
\begin{aligned}
\frac{g''}{g} \geq &\left(1 - \sqrt{\frac{2gR}{\varkappa}}\right) \left(1 - \frac{\delta}{\|b\|_4^4} - \frac{R\|\nabla b^2\|_\infty / r}{\|b\|_4^4}\right) \\
&\times (1-\varepsilon)\left(1 - 2R\|\nabla b^2\|_\infty / r\delta\right) \left(1 - (n-2)\frac{\pi R^2}{r^2}\|b\|_\infty^2\right) \\
&\times \left(1 - \frac{3n}{\varepsilon \widetilde{e}^{\perp}} \frac{ar^2}{R^3} \frac{1}{1-(aR_0/R)^3} \left[1 - \frac{n^2}{\varepsilon \widetilde{e}^{\perp}} \frac{a}{R} 3\pi \|b\|_4^4 \frac{1}{1-(aR_0/R)^3}\right]\right) .
\end{aligned}
$$

It remains to choose the free parameters R, δ, $\varkappa$, ε and η. If we choose

$$
R = n\left(\frac{na}{r}\right)^{1/4} \quad , \quad \delta = \varepsilon = \eta = \left(\frac{na}{r}\right)^{1/8} \quad , \quad \varkappa = \frac{1}{n}\left(\frac{na}{r}\right)^{5/12} ,
$$

Eq. (4.11) implies

$$
\left\langle \Phi \left| H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^{\perp}}{r^2} \right| \Phi \right\rangle \geq \left(1 - D\left(\frac{na}{r}\right)^{1/8} - Dn^2\left(\frac{na}{r}\right)^{3/8}\right) \left\langle f \left| H_{1\mathrm{d}}^{n,\ell,g} \right| f \right\rangle
$$

for some constant $D > 0$. In particular, if $P_r^{\perp}$ denotes the projection onto the function $\prod_{k=1}^n b_r(\mathbf{x}_k^{\perp}) \in L^2(\mathbb{R}^{2n})$, we have the operator inequality

$$
H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^{\perp}}{r^2} \geq (1-\eta_{\mathrm{L}})\, P_r^{\perp} \otimes H_{1\mathrm{d}}^{n,\ell,g}
$$

in the sense of quadratic forms. (Here, η_{L} is defined in (2.4).)

Recall that that the gap above the ground state energy of $-\Delta^\perp + r^{-2}V^\perp(\mathbf{x}^\perp/r)$ is $\widetilde{e}^\perp/r^2$. Hence, using the positivity of both $V^\parallel$ and v,

$$H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^\perp}{r^2} \geq \frac{\widetilde{e}^\perp}{r^2}\left(\mathbb{1} - P_r^\perp\right) \otimes \mathbb{1} .$$

In particular,

$$H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^\perp}{r^2} \geq (1-\gamma)\,(1-\eta_{\mathrm{L}})\,P_r^\perp \otimes H_{1\mathrm{d}}^{n,\ell,g} + \gamma\frac{\widetilde{e}^\perp}{r^2}\left(\mathbb{1} - P_r^\perp\right) \otimes \mathbb{1} \tag{4.12}$$

for any $0 \leq \gamma \leq 1$. If we choose $\gamma = (r^2/\widetilde{e}^\perp)E_{1\mathrm{d}}^k(n,\ell,g)$, the lowest k eigenvalues of the operator on the right side of (4.12) are given by $(1-\gamma)(1-\eta_{\mathrm{L}})E_{1\mathrm{d}}^i(n,\ell,g)$, $1 \leq i \leq k$. This implies the lower bound (2.3) of Theorem 1.

5 Estimates on Eigenfunctions

In this final section, we shall show how the estimates in the previous two sections can be combined to yield a relation between the eigenfunctions Ψ_k of $H_{3\mathrm{d}}^{n,r,\ell,a}$ and the eigenfunctions ψ_k of $H_{1\mathrm{d}}^{n,\ell,g}$. We start with the following simple Lemma.

LEMMA 2. *Let H be a non-negative Hamiltonian on a Hilbertspace $\mathcal{H}$, with eigenvalues $0 \leq E_1 \leq E_2 \leq \ldots$ and corresponding (orthonormal) eigenstates ψ_i. For $k \geq 1$, let f_i, $1 \leq i \leq k$, be orthonormal states in $\mathcal{H}$, with the property that $\langle f_i|H|f_i\rangle \leq \eta E_i$ for some $\eta > 1$. If $E_{k+1} > E_k$, then*

$$\sum_{i,j=1}^k |\langle f_i|\psi_j\rangle|^2 \geq k - \frac{(\eta-1)\sum_{i=1}^k E_i}{E_{k+1}-E_k} .$$

Proof. Let P_k be the projection onto the first k eigenfunctions of H. Then $H \geq HP_k + E_{k+1}(1-P_k)$. By the variational principle,

$$\sum_{i=1}^k \langle f_i|HP_k + E_k(1-P_k)|f_i\rangle \geq \sum_{i=1}^k E_i ,$$

and hence

$$\sum_{i=1}^k \langle f_i|H|f_i\rangle \geq \sum_{i=1}^k E_i + (E_{k+1}-E_k)\sum_{i=1}^k \langle f_i|1-P_k|f_i\rangle .$$

By assumption, $\langle f_i|H|f_i\rangle \leq \eta E_i$, implying the desired bound

$$\sum_{i=1}^k \langle f_i|P_k|f_i\rangle \geq k - \frac{(\eta-1)\sum_{i=1}^k E_i}{E_{k+1}-E_k} .$$

■

COROLLARY 3. *Assume in addition that $E_{l+1} > E_l$ for some $1 \le l < k$. Then*

$$\sum_{i,j=l+1}^{k} |\langle f_i|\psi_j\rangle|^2 \ge k - l - \frac{(\eta-1)\sum_{i=1}^{k} E_i}{E_{k+1}-E_k} - \frac{(\eta-1)\sum_{i=1}^{l} E_i}{E_{l+1}-E_l}\,.$$

Proof. Applying Lemma 2 and using the orthonormality of the ψ_i and the f_i, we have

$$\begin{aligned}\sum_{i,j=l+1}^{k} |\langle f_i|\psi_j\rangle|^2 &= \sum_{i,j=1}^{k} |\langle f_i|\psi_j\rangle|^2 + \sum_{i,j=1}^{l} |\langle f_i|\psi_j\rangle|^2 \\ &\quad - \sum_{i=1}^{l}\sum_{j=1}^{k} |\langle f_i|\psi_j\rangle|^2 - \sum_{i=1}^{k}\sum_{j=1}^{l} |\langle f_i|\psi_j\rangle|^2 \\ &\ge k - \frac{(\eta-1)\sum_{i=1}^{k} E_i}{E_{k+1}-E_k} + l - \frac{(\eta-1)\sum_{i=1}^{l} E_i}{E_{l+1}-E_l} - 2l\,.\end{aligned}$$

$\blacksquare$

Let again $P_r^\perp$ denote the projection onto the function $\prod_{k=1}^n b_r(\mathbf{x}_k^\perp) \in L^2(\mathbb{R}^{2n})$, and recall inequality (4.12), which states that

$$H_{3\mathrm{d}}^{n,r,\ell,a} - \frac{ne^\perp}{r^2} \ge (1-\gamma)\,(1-\eta_{\mathrm{L}})\,P_r^\perp \otimes H_{1\mathrm{d}}^{n,\ell,g} + \gamma\frac{\widetilde{e}^\perp}{r^2}\left(\mathbb{1} - P_r^\perp\right)\otimes \mathbb{1} \tag{5.1}$$

for any $0 \le \gamma \le 1$. As already noted, the choice

$$\gamma = (r^2/\widetilde{e}^\perp)E_{1\mathrm{d}}^k(n,\ell,g)$$

implies that the lowest k eigenvalues of the operator on the right side of (5.1) are given by $(1-\gamma)(1-\eta_{\mathrm{L}})E_{1\mathrm{d}}^i(n,\ell,g)$, $1 \le i \le k$. Here, we have to assume that $\widetilde{e}^\perp/r^2 \ge E_{1\mathrm{d}}^k(n,\ell,g)$ in order for γ not to be greater than one. The corresponding eigenfunctions are

$$\psi_i(z_1,\ldots,z_n)\prod_{k=1}^n b_r(\mathbf{x}_k^\perp)\,.$$

We can now apply Corollary 3, with H equal to the operator on the right side of (5.1), $f_i = \Psi_i$, $l = k_i - 1$ and $k = k_{i+1}-1$. Since $E_{3\mathrm{d}}^k(n,r,\ell,a) \le E_{1\mathrm{d}}^k(n,\ell,g)/(1-\eta_{\mathrm{U}})$ by Theorem 1(b), we conclude that

$$\begin{aligned}&\sum_{k=k_i}^{k_{i+1}-1}\sum_{l=k_i}^{k_{i+1}-1}\left|\left\langle \Psi_k \,\middle|\, \psi_l \prod_{i=1}^n b_r(\mathbf{x}_k^\perp)\right\rangle\right|^2 \\ &\ge k_{i+1} - k_i - \left(\frac{1}{(1-\gamma)(1-\eta_{\mathrm{U}})(1-\eta_{\mathrm{L}})} - 1\right)\left[\frac{\sum_{i=1}^{k_{i+1}-1} E_{1\mathrm{d}}^i}{E_{1\mathrm{d}}^{k_{i+1}} - E_{1\mathrm{d}}^{k_i}} + \frac{\sum_{i=1}^{k_i-1} E_{1\mathrm{d}}^i}{E_{1\mathrm{d}}^{k_i} - E_{1\mathrm{d}}^{k_i-1}}\right]\end{aligned}$$

as long as $\gamma < 1$, $\eta_{\rm U} < 1$ and $\eta_{\rm L} < 1$. Here, we used the notation $E^i_{\rm 1d} = E^i_{\rm 1d}(n,\ell,g)$ for short. Since

$$\sum_{l=k_i}^{k_{i+1}-1} \left| \left\langle \Psi_k \middle| \psi_l \prod_{i=1}^n b_r(\mathbf{x}_k^\perp) \right\rangle \right|^2 \leq 1$$

for every k, this implies Theorem 2.

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